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"CAPUTO-FABRIZIO FRACTIONAL DERIVATIVES OF AGE-STRUCTURED DIPHTHERIA INFECTION MODEL WITH LAPLACE ADOMIAN DECOMPOSITION ANALYSIS "

CAPUTO-FABRIZIO FRACTIONAL DERIVATIVES OF AGE-STRUCTURED DIPHTHERIA INFECTION MODEL WITH LAPLACE ADOMIAN DECOMPOSITION ANALYSIS.

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Abstract:

Diphtheria is a bacterial infectious disease that can lead to severe complications and even deaths. This work presents the **Caputo-Fabrizio** Fractional derivatives of the aged-structured deterministic model of diphtheria infection. The existence and the uniqueness of the solution of the model are investigated and established using the contraction principle. The stability of the model is investigated with the help of the well-known Ulem-Hyers and the generalized Ulem-Hyers theorems. Analyzing the model using the Laplace Adomian Decomposition Methods,the system's analytical solution, in the form of an infinite series that converges quickly to it exact value is obtained.

KeyWords: Diphtheria, **Caputo-Fabrizio**, Adomian Decomposition, aged-structured, contraction principle

Introduction

1.1 Introduction

Diphtheria is one of the respiratory diseases raphaging the population in recent time. It is a bacterial (Corynebacteriumdiptheriae) infectious disease that can lead to severe complications such as respiratory failure, heart problems and even deaths if it is not detected early. This infection that mostly affects the throat and the nose can be prevented by vaccination. Case-fatality occurs only in places where there is poor sanitation condition and inadequate vaccination coverage as a result of low resources [1,2,3]

Diphtheria is a highly contagious infection that spreads primarily through person-to-person contact via respiratory droplets(coughing, sneezing or spitting) or direct contact with infected skin lesions or any material (like clothe) that has been in contact with the bacteria. It is possible to get diphtheria more than once. Anyone who is not protected by diphtheria vaccine and comes in close contact with diphtheria is susceptible. Some of the symptoms/signs of diphtheria are: throat pain, weakness/ fatigue, fever, swollen neck glands, problems breathing due to tissues obstructing nose and throat, nerve, kidney or heart problems (if the bacteria enters the blood stream). The incubation period is one to ten days after exposure(4, 5, 6,7,8)

To effectively contend the spread of Diphtheria, different control measures, such as isolation of patient, maintenance of one meter between patients, keeping patient care areas with good ventilation, the use mask that is medically prepared and cover any wound/lesions on patient's body by patient who may have to move out of the isolation areas, population subgroup such as young children under five years of age, school children, elderly who are at greater risk and have close contact with diphtheria infection and health workers should highly prioritized with treatment and vaccination, epidemiological surveillance ensuring early detection of diphtheria outbreak, administering antitoxin to neutralize the toxin and antibiotics to kill the bacteria, reducing complication and mortality should be implemented [6,7,8].

Understanding, describing and analyzing the dynamics of infectious diseases [9, 10, 11, 12, 13, 14, 15,16, 17,18] have been key in guiding decisions and policies in public health system.

In recent time, mathematicians and epidemiologist have demonstrated great effort in understanding and describing the dynamics of diphtheria infection.

[19], presented the mathematical model of diphtheria diseases that categorizes the individuals based on susceptibility, vaccination, infected and recovery status. The stability of the system wa confirmed, the basic reproductive ratio was calculated. They also converted the deterministic model into caputo-Fabrizio fractional order model, analyzed it for existence and uniqueness of solution using appropriate principle. Adomian Decomposition method was applied for numerical solution of the model.

In the research under consideration, we seek to transform the aged-structured deterministic model of diphtheria infection [8,9] into the Caputo-Fractional order of aged-structured model of diphtheria infection. The existence and the uniqueness of the solution of the model shall be investigated and established using the contraction principle. The stability of the model shall be investigated with the help of the well-known Ulem-Hyers and the generalized Ulem-Hyers theorems. Analyzing the model using the Laplace Adomian Decomposition Methods, the system's analytical solution, in the form of an infinite series that converges quickly to it exact value shall be obtained.

2.0 MODEL FORMULATION

2.1 ASSUMPTIONS

1. The control of diphtheria is based on primary prevention of disease by ensuring high population immunity of the infant ((0-1 year) and school children by vaccination
2. Isolation of detected cases (that is confirmed cases are not allowed to interact with the population freely.
3. Epidemiological surveillance ensuring early detection through contact tracing is carried out
4. Secondary prevention spread by the rapid investigation of close contacts to ensure prompt treatment of those infected
5. The total population of human at time t under consideration denoted by N_h is split into mutually exclusive sub-population of S_1 Susceptible infant at time t (0-1years), S_2 , Susceptible school children population at time t, V, Vaccination population at time t, E, Exposed population at time t , I_1, Asymptomatic infection population at time t, I_2., Symptomatic infection population at time t, R., Recovered population at time t. I_D., Detected infectious humans at time t(Asymptomatic and symptomatic) population through testing,

$$N_h = S_1 + S_2 + V + E + I_1 + I_2 + R + I_D$$

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2.2 STATE VARIABLES

N_h −The total population of human at time t under consideration

S_1 − Susceptible infantat time t (0-1years),

S_2 − Susceptible school children populationat time t,

V −Vaccination populationat time t,

E − Exposed populationat time ,

I_1. − Asymptomatic infection populationat time t,

I_2. − Symptomatic infection populationat time t,

R −Recovered populationat time t.

I_D. − Detected infectious humansat time t(Asymptomatic and symptomatic) population through testing,

2.3 PARAMETERS

$\sigma_1 - progress\ rate\ from\ exposed\ to\ I_1$, Asymptomatic infection population

$\sigma_2 - progress\ rate\ efrom\ exposed\ to\ I_2$, symptomatic infection population

$\delta - fraction\ of\ new\ infection\ that\ are\ \ I_1$, Asymptomatic

$1 - \delta - fraction\ of\ new\ infection\ that\ are\ \ I_2$, symptomatic

$\beta_1 - effective\ transmission\ rate\ from\ S_1$

$\beta_2 - effective\ transmission\ rate\ from\ S_2$

$\tau_1 - vaccination\ coverage\ \ for\ \ S_1$

$\tau_2 - vaccination\ coverage\ \ for\ \ S_2$

$\varepsilon - vaccine\ efficacy$

$\rho - maturity\ rate\ from\ \ S_1\ to\ S_2$

$b -$ *per capita birth rate of humans*

$\varepsilon\tau_1 -$ *immunization rate in* S_1

$\varepsilon\tau_2 -$ *immunization rate in* S_2

$\omega -$ *rate of vaccination*

$\theta_1 -$ *detection rate (via contact tracing) for* S_1

$\theta_2 -$ *detection rate (via contact tracing) for* S_2

$\eta -$ *death due to infection*

$\alpha -$ *natural death rate*

$$\lambda_1(t) = \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}$$

$$\lambda_2(t) = \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}$$

2.4 MODEL EQUATIONS

Using the above described state variables and parameters together with the schematic diagram in figure 1, the model of the diphtheria infection transmission dynamics results in the following system of deterministic non-linear first order differential equations

$$\frac{dS_1}{dt} = bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1 \qquad (2.1)$$

$$\frac{dS_2}{dt} = \rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2 \qquad (2.2)$$

$$\frac{dV}{dt} = \varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \alpha V \qquad (2.3)$$

$$\frac{dE}{dt} = \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \sigma_1 \delta E - \sigma_2(1-\delta)E - \alpha E \qquad (2.4)$$

$$\frac{dI_1}{dt} = \sigma_1 \delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1 \qquad (2.5)$$

$$\frac{dI_2}{dt} = \sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2 \qquad (2.6)$$

$$\frac{dI_D}{dt} = \theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D \qquad (2.7)$$

$$\frac{dR}{dt} = \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R \qquad (2.8)$$

2.5 FRACTIONAL ORDER MODEL

The Caputo-Fabrizio order derivatives of (2.1) –(2.8) is given as follows

$${}_0^{CF}D_t^{\xi} S_1(t) = bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1 \qquad (3.1)$$

$${}_0^{CF}D_t^{\xi} S_2(t) = \rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2 \qquad (3.2)$$

$${}_0^{CF}D_t^{\xi} V(t) = \varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \alpha V \qquad (3.3)$$

$${}_0^{CF}D_t^{\xi} E(t) = \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \sigma_1\delta E - \sigma_2(1-\delta)E - \alpha E \qquad (3.4)$$

$${}_0^{C}D_t^{\xi} I_1(t) = \sigma_1\delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1 \qquad (3.5)$$

$${}_0^{CF}D_t^{\xi} I_2(t) = \sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2 \qquad (3.6)$$

$${}_0^{CF}D_t^{\xi} I_D(t) = \theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D \qquad (3.7)$$

$${}_0^{CF}D_t^{\xi} R(t) = \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R \qquad (3.8)$$

With the initial conditions,

$$S_1(0) = S_{10},\ S_2(t) = S_{20}, V(t) = V_0, E(t) = E_0,\ I_1(t) = I_{10}, I_2(t) = I_{20},, I_D(t) = I_{D0}\ R(t) = R_0$$

$$S_1(t) + S_2(t) + V(t) + E(t) + I_1(t) + I_2(t) + I_D(t) + R(t) = 1$$

${}_0^{CF}D_t^{\xi}$ Represents the **Caputo-Fabrizio** fractional derivative of order $\xi \epsilon [0, 1]$

3.0 ANALYSIS OF THE MODEL

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3.1 EXISTENCE AND UNIQUENESS OF SOLUTION

We shall use some basic fixed point theorem to establish the existence and uniqueness of the solution of (3.1) –(3.8)

$$ {}_{0}^{CF}D_t^{\xi}\psi(t) = \mathcal{L}(t,\psi(t) \qquad (3.9)$$

$$\psi(0) = \psi_0$$

$\psi(t) = \left(S_1(t), S_2(t), V(t), E(t), I_1(t),\ I_2(t),\ I_D(t), R(t)\right)^T \in \Re^8\ for\ t \in [0.T_{max}]$ denotes the state of the model and $\mathcal{L}$ represent the continuous vector given below

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \mathcal{L}_5 \\ \mathcal{L}_6 \\ \mathcal{L}_7 \\ \mathcal{L}_8 \end{pmatrix} = \begin{pmatrix} bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1 \\ \rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2 \\ \varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \omega V - \alpha V \\ \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \sigma_1 \delta E - \sigma_2(1-\delta)E - \alpha E \\ \sigma_1 \delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1 \\ \sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2 \\ \theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D \\ \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R \end{pmatrix} \quad (3.10)$$

The initial condition of the variable of the model is denoted by $\big(S_1(0), S_2(0), V(0), E(0), I_1(0),$ $I_2(0),\ I_D(0), R(0)\big)^T$

Where $\psi(0) = \left(S_1(0), S_2(0), V(0), E(0), I_1(0),\ I_2(0),\ I_D(0), R(0)\right)^T$ *and* , $\psi_0 = (S_{10}, S_{20}, V_0, E_0, I_{10},$ $I_{20},\ I_{D0}, R_0)^T$

In addition, we define $\mathcal{L}: [0.T_{max}] \times \Re^8 \rightarrow \Re^8$ is said to satisfy Lipschitz condition in the second argument, if we have:

$$\left\|\mathcal{L}(t,\psi_1) - \mathcal{L}(t,\psi_2)\right\| \le M\left\|\psi_1 - \psi_2\right\| \ \forall\ t \in [0, T_{max}]\ \forall\ \psi_1, \psi_2\ \in \mathbb{R}^8\ where\ M > 0, T_{max}\ is\ final\ time \ldots\ldots\ldots\ldots\ldots\ldots (3.11)$$

The existence of a unique solution to the model (3.1) - (3.8)is established in the following theorem:

Theorem 3.1.

There exists a unique solution to the initial value problem (3.9) on $C([0, T_{max}], \mathbb{R}^8)$, provided that (3.11) and

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$$\left(\frac{2(1-\xi)M}{(2-\xi)\mathcal{F}(\xi)}+\frac{2\xi M}{(2-\xi)\mathcal{F}(\xi)}T_{max}\right)<1 \ldots\ldots\ldots\ldots\ldots\ldots..(3.12)$$

are satisfied.

Proof:

If we apply the **Caputo-Fabrizio** fractional integral on each sides of (3.9), then we have the **Caputo-Fabrizio** time-fractional integral of the function $\psi(t)$ of order ξ is defined by

$$\psi(t)= \psi_0+\frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)}\psi(r,t)+\frac{2\xi}{(2-\xi)\mathcal{F}(\xi)}\int_0^t \mathcal{L}(\tau,\psi(\tau)d\tau \qquad (3.13)$$

Let us defined the operator K: $C([0,T_{max}],\mathbb{R}^8) \rightarrow C([0,T_{max}],\mathbb{R}^8)$ by

$$\mathrm{K}[\psi](t)=V(t), \qquad \psi,\mathrm{V}\in C\big[[0,T_{max}],\mathbb{R}^8\big] \qquad (3.14)$$

Where

$$V(t)= \psi_0+\frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)}V(r,t)+\frac{2\xi}{(2-\xi)\mathcal{F}(\xi)}\int_0^t \mathcal{L}(\tau,V(\tau)d\tau$$

The supremum norm on $C\big[[0,T_{max}],\mathbb{R}^8\big]$ is given by:

$$\|V(t)\|= \sup_{t\in[0,T_{max}]}\|V(t)\| , \qquad \forall\, V\in C([0,T_{max}],\mathbb{R}^8)$$

Clearly, $C([0,T_{max}],\mathbb{R}^8)$ equipped with $\|.\|$ is a Banach space.

Suppose, $\wp$ is the fixed point of the operator

K: $C([0,T_{max}],\mathbb{R}^8) \rightarrow C([0,T_{max}],\mathbb{R}^8)$, then $\wp$ becomes the solution of the initial value problem (3.9), and

$$\mathrm{K}\wp(t)=\wp(t)$$

Where

$$\wp(t)= \psi_0+\frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)}\mathcal{L}(t,\wp(\mathrm{t}))+\frac{2\xi}{(2-\xi)\mathcal{F}(\xi)}\int_0^t \mathcal{L}(\tau,\wp(\tau)d\tau$$

Consider

$$\|\mathrm{K}[V](t)-\mathrm{K}[\wp](t)\| = \Bigg\|\psi_0+\frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)}\mathcal{L}(t,\mathrm{V(t)})+\frac{2\xi}{(2-\xi)\mathcal{F}(\xi)}\int_0^t \mathcal{L}(\tau,V(\tau)d\tau - \left[\psi_0+\frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)}\mathcal{L}(t,\wp(\mathrm{t}))+\frac{2\xi}{(2-\xi)\mathcal{F}(\xi)}\int_0^t \mathcal{L}(\tau,\wp(\tau)d\tau\right]\Bigg\|$$

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$$\begin{aligned}
&\|\mathrm{K}[V](t)-\mathrm{K}[\wp](t)\| \\
&\qquad \le \left\| \frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)} \frac{1}{\Gamma(\xi)} \Big(\mathcal{L}\big(t, \mathrm{V(t)}\big) - \mathcal{L}\big(t, \wp(\mathrm{t})\big)\Big) \right. \\
&\qquad \left. + \frac{2\xi}{(2-\xi)\mathcal{F}(\xi)} \int_0^t [\mathcal{L}(\tau, V(\tau) - \mathcal{L}(\tau, \wp(\tau)] d\tau \right\| \\
&\le \frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)} \left\| \Big(\mathcal{L}\big(t, \mathrm{V(t)}\big) - \mathcal{L}\big(t, \wp(\mathrm{t})\big)\Big) \right\| \\
&\qquad + \frac{2\xi}{(2-\xi)\mathcal{F}(\xi)} \left\| \int_0^t [\mathcal{L}(\tau, V(\tau) - \mathcal{L}(\tau, \wp(\tau)] d\tau \right\| \qquad (3.15)
\end{aligned}$$

Since the operator $\mathcal{L}$satisfies the Lipschitz condition (eq. 3.11), we have that

$$\|\mathrm{K}[V](t)-\mathrm{K}[\wp](t)\| \le \frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)} M \|\mathrm{V(t)} - \wp(\mathrm{t})\| + \frac{2\xi M}{(2-\xi)\mathcal{F}(\xi)} \int_0^t \|\mathrm{V(t)} - \wp(\mathrm{t})\| d\tau$$

$$\begin{aligned}
&\|\mathrm{K}[V](t)-\mathrm{K}[\wp](t)\| \\
&\qquad \le \frac{2(1-\xi)}{(2-\xi)\mathcal{F}(\xi)} M \sup_{t\in[0,T_{max}]} \|\mathrm{V(t)} - \wp(\mathrm{t})\| \\
&\qquad + \frac{2\xi M}{(2-\xi)\mathcal{F}(\xi)} \int_0^t \sup_{t\in[0,T_{max}]} \|\mathrm{V(t)} - \wp(\mathrm{t})\| d\tau \qquad (3.16)
\end{aligned}$$

$$\le \frac{2(1-\xi)M}{(2-\xi)\mathcal{F}(\xi)} + \frac{2\xi M \int_0^t d\tau}{(2-\xi)\mathcal{F}(\xi)} \|\mathrm{V} - \wp\|$$

$$\le \frac{2(1-\xi)M}{(2-\xi)\mathcal{F}(\xi)} + \frac{2\xi M T_{max}}{(2-\xi)\mathcal{F}(\xi)} \|\mathrm{V} - \wp\|$$

Thus if the condition (3.12) holds then,

$$\|\mathrm{K}[V](t)-\mathrm{K}[W](t)\| \le \frac{2(1-\xi)M}{(2-\xi)\mathcal{F}(\xi)} + \frac{2\xi M T_{max}}{(2-\xi)\mathcal{F}(\xi)} \|\mathrm{V} - \mathrm{W}\|$$

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Hence, the operator K becomes a contraction. Therefore K has a unique fixed point which is a solution to the initial value problem (3.9) and hence a solution to the system ((3.1)-(3.8)).

ULAM-HYERS STABILITY

The Ulam-Hyers (UH) stability and generalized UH stability [20,21]for the fractional system (3.1) –(3.8) using the Caputo-Fabrizio operator is discussed in this section. Let S = $C([0,T_{max}]:\mathbb{R}^8)$ be the space of all continuous functions from $[0,T_{max}]$ to $\mathbb{R}^8$, endowed with the norm: $\|\psi\| = \sup_{t\in[0,T_{max}]}\|\psi\|$,

Consider

$$ {}^{CF}_{0}D^{\xi}_{t}\psi(t) = \mathcal{L}(t.\psi(t)) \qquad (3.17) $$

$$ \psi(t) = \psi_0 $$

Also let $\varepsilon > 0$. Consider the following inequality:

$$ \left\| {}^{CF}_{0}D^{\xi}_{t}\overline{\psi}(t) - \mathcal{L}(t.\overline{\psi}(t)) \right\| \le \varepsilon, t\,\epsilon\Im, \quad \varepsilon = \max(\varepsilon_i)^T, i = 1,2,3,\ldots,8, \overline{\psi} \in S \qquad (3.18) $$

Remark 3.1. "A function $\psi \in S$ satisfies the inequality (3.18) if and only if there exists a function $p \in S$, having the following properties;

$$ (i)|p(t)| \le \varepsilon, p = \max(p_j), t\,\epsilon\Im \text{ (ii)} {}^{C}_{0}D^{\xi}_{t}\overline{\psi}(t) = \mathcal{L}(t.\overline{\psi}(t)) + p(t), t\,\epsilon\Im $$

Definition 3.1. The fractional model ((3.1)-(3.8)) or the transformed system (3.17) is UH stable if for every ε > 0 there exists k > 0, such that for any solution φ ∈ S of the inequality (3.18), there exists a unique solution $\psi \in S$, of the fractional system (3.17) such that the following inequality is satisfied:

$$ \|\overline{\psi}(t) - \psi(t)\| \le k\varepsilon, t\,\epsilon\Im k = \max(k_j), j = 1,2,3,\ldots,8 \qquad (3.19) $$

Where

$$ \psi(t) = (S_1(t), S_2(t), V(t), E(t), I_1(t), I_2(t), \ I_D(t), R(t))^T $$
$$ \bar{\psi}(t) = (\bar{\bar{S}}_1(t), \bar{S}_2(t), \bar{V}(t), \bar{E}(t), \bar{I}_1(t), \bar{I}_2(t), \ \bar{I}_D(t), \bar{R}(t))^T $$
$$ \psi(0) = (S_1(0), S_2(0), V(0), E(0), I_1(0), I_2(0), I_D(0), R(0))^T $$

Definition 3.2. The model system (3.17) is generalized UH stable if there exists a continuous function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$satisfying $\psi(0) = 0$, such that for any solution $\psi \in S$ of system (3.18), there exists a unique solution $\psi \in S$ such that the following inequality is satisfied:

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$$\|\overline{\psi}(t)-\psi(t)\| \le \psi(\varepsilon), t\, \epsilon \Im \psi = \max\left(\psi_j\right)^T, j = 1,2,3,\dots,8 \qquad (3.20)$$

Theorem 3.4. If $\Phi \in$ E satisfies the system (3.17), then we have the following:

$$\left|\overline{\psi}(t)-\overline{\psi}_0(t)-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}u(r,t)+\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t \mathcal{L}\big(\tau.\overline{\psi}(\tau)\big)d\tau\right| \le \Omega\varepsilon$$

$$where\ \Omega = \frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}+\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t d\tau \ \ (3.19)$$

Proof:

Using remark 3.1(ii) ${}_0^{CF}D_t^{\xi}\overline{\psi}(t) = \mathcal{L}\big(t.\overline{\psi}(t)\big)+p(t), t\, \epsilon\Im$, which on applying the Caputo-Fabrizio integral gives

$$\overline{\psi}(t) = \overline{\psi}_0(t)+\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}u(r,t)+\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t \mathcal{L}\big(\tau.\overline{\psi}(\tau)\big)d\tau+\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}p(t) + \frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t p(\tau)d\tau$$

By rearranging, applying norm on both sides and using remark 4.1 (i), it follows that;

$$\left|\overline{\psi}(t)-\overline{\psi}_0(t)-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}u(r,t)-\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t \mathcal{L}\big(\tau.\overline{\psi}(\tau)\big)d\tau\right| \le \frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}p(t)+\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t |p(\tau)|d\tau \le \Omega\varepsilon$$

Theorem 3.5. Suppose$\mathcal{L}: \Im \times \mathbb{R}^8 \to \mathbb{R}^8$ satisfies the Lipschitz condition, with Lipschitz constant $M > 0$ and$(1-\Omega)\ M > 0$, then the model (3.17) is generalized UH stable.

Proof:

Suppose that $\psi \in$S satisfies the inequality in (3.18) and $\psi \in$S is a unique solution of (3.17). Then $\forall\varepsilon > 0$; $t\epsilon[0, T_{max}]$, we have

$$|\overline{\psi}(t)-\psi(t)|$$

$$=\max_{t\epsilon[0,T_{max}]}\left|\overline{\psi}(t)-\overline{\psi}_0(t)-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}\overline{\psi}(r,t)-\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t\mathcal{L}(\tau.\overline{\psi}(\tau))d\tau\right|$$

$$\leq\max_{t\epsilon[0,T_{max}]}\left|\overline{\psi}(t)-\overline{\psi}_0(t)-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}\overline{\psi}(r,t)-\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t\mathcal{L}(\tau.\overline{\psi}(\tau))d\tau\right|$$

$$+\max_{t\epsilon[0,T_{max}]}\left\|\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}\overline{\psi}(r,t)-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}\psi(r,t)\right\|$$

$$+\max_{t\epsilon[0,T_{max}]}\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t\left|\mathcal{L}(\tau.\overline{\psi}(\tau))-\mathcal{L}(\tau.\psi(\tau))\right|d\tau$$

$$\leq\max_{t\epsilon[0,T_{max}]}\left|\overline{\psi}(t)-\overline{\psi}_0(t)-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}\overline{\psi}(r,t)-\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t\mathcal{L}(\tau.\overline{\psi}(\tau))d\tau\right|$$

$$+\max_{t\epsilon[0,T_{max}]}\frac{2(1-\alpha)}{(2-\alpha)\mathcal{F}(\alpha)}|\overline{\psi}(r,t)-\psi(r,t)|$$

$$+\max_{t\epsilon[0,T_{max}]}\frac{2\alpha}{(2-\alpha)\mathcal{F}(\alpha)}\int_0^t|\overline{\psi}(\tau)-\psi(\tau)|\,d\tau$$

$$\leq\varepsilon\Omega+\Omega\mathcal{M}|\overline{\psi}-\psi|$$

Thus, we have $\|\overline{\psi}-\psi\|\leq k\varepsilon, where\ k=\frac{\Omega}{1-\Omega\mathcal{M}}$ (3.20)

Hence equating $\psi(\varepsilon)=k\varepsilon$, so that $\psi(0)=0$we conclude that the model (3.16),is both UH and generalized UH stable.

4.0 Iterative schemes involving the Caputo fractional operators

This section, we shall study an iterative scheme using the Caputo-Fabrizio fractional derivative. **We seek to derive explicit expressions for the unknown functions,$S_1(t)$, $S_2(t), V(t), E(t), I_1(t), I_2(t), R(t), I_D(t)$ using series representation approach based on Laplace Adomian Decomposition Method. The system is transformed to algebraic equations by the application of Laplace transform to the** Caputo-Fabrizio **fractional order derivative (3.1) – (3.8). Laplace Adomian Decomposition Method empowers us to construct a convergent series solution for**

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$S_1(t),\ S_2(t), V(t), E(t), I_1(t), I_2(t), R(t), I_D(t)$which can be evaluated numerically to obtain accurate approximations.

For the solution of the model (3.1)-(3.8), we shall adopt the Laplace Adomian Decomposition method. Applying the Laplace transform of the Caputo-Fabrizio fractional operator to both sides of the system (3.1)-(3.8), we have

$$\mathcal{L}\left\{{}^{CF}_{0}D^{\xi}_{t} S_1(t)\right\} = \mathcal{L}\left\{bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1\right\} \qquad (4.1.1)$$

$$\mathcal{L}\left\{{}^{CF}_{0}D^{\xi}_{t} S_2(t)\right\} = \mathcal{L}\left\{\rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2\right\} \qquad (4.1.2)$$

$$\mathcal{L}\left\{{}^{CF}_{0}D^{\xi}_{t} V(t)\right\} = \mathcal{L}\{\varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \omega V - \alpha V\} \qquad (4.1.3)$$

$$\mathcal{L}\left\{{}^{CF}_{0}D^{\xi}_{t} E(t)\right\} = \mathcal{L}\left\{\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \sigma_1\delta E - \sigma_2(1-\delta)E - \alpha E\right\} \qquad (4.1.4)$$

$$\mathcal{L}\left\{{}^{CF}_{0}D^{\xi}_{t} I_1(t)\right\} = \mathcal{L}\{\sigma_1\delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1\} \qquad (4.1.5)$$

$$\mathcal{L}\left\{{}^{CF}_{0}D^{\xi}_{t} I_2(t)\right\} = \mathcal{L}\{\sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2\} \qquad (4.1.6)$$

$$\mathcal{L}\left\{{}^{C}_{0}D^{\xi}_{t} R(t)\right\} = \mathcal{L}\{\gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R\} \qquad (4.1.7)$$

$$\mathcal{L}\left\{{}^{CF}_{0}D^{\xi}_{t} I_D(t)\right\} = \mathcal{L}\{\theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D\} \qquad (4.1.8)$$

Using the property of Laplace transform for Caputo-Fabrizio fractional derivatives, we obtain

Following the definition of Laplace transform for the Caputo-Fabrizio derivative,
(Laplace transform of the Caputo-Fabrizio derivative of functions)

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The Laplace transform of the Caputo-Fabrizio derivative is given by

$$\mathcal{L}\{{}^{CF}_{0}D^{\alpha}_{t}u(r,t)\}(s) = \frac{(2-\alpha)\mathcal{F}(\alpha)}{2}\frac{s\mathcal{L}\{u(r,t)\} - u(r,0)}{s+\alpha(1-s)}$$

$$\frac{s\mathcal{L}\{S_1(t)\} - S_1(0)}{s+\xi(1-s)} = \mathcal{L}\left\{bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1\right\} \quad (4.2.1)$$

$$\frac{s\mathcal{L}\{S_2(t)\} - S_2(0)}{s+\xi(1-s)} = \mathcal{L}\left\{\rho S_1 - \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}S_2 - \varepsilon\tau_2 S_2 - \alpha S_2\right\} \quad (4.2.2)$$

$$\frac{s\mathcal{L}\{V(t)\} - V(0)}{s+\xi(1-s)} = \mathcal{L}\{\varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \omega V - \alpha V\} \quad (4.2.3)$$

$$\frac{s\mathcal{L}\{E(t)\} - E(0)}{s+\xi(1-s)} = \mathcal{L}\left\{\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}S_2 - \sigma_1\delta E - \sigma_2(1-\delta)E - \alpha E\right\} \quad (4.2.4)$$

$$\frac{s\mathcal{L}\{I_1(t)\} - I_1(0)}{s+\xi(1-s)} = \mathcal{L}\{\sigma_1\delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1\} \quad (4.2.5)$$

$$\frac{s\mathcal{L}\{I_2(t)\} - I_2(0)}{s+\xi(1-s)} = \mathcal{L}\{\sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2\} \quad (4.2.6)$$

$$\frac{s\mathcal{L}\{I_D(t)\} - I_D(0)}{s+\xi(1-s)} = \mathcal{L}\{\theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D\} \quad (4,2.7)$$

$$\frac{s\mathcal{L}\{R(t)\} - R(0)}{s+\xi(1-s)} = \mathcal{L}\{\gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R\} \quad (4.2.8)$$

$$\mathcal{L}\{S_1(t)\} = \frac{S_1(0)}{s} + \frac{s+\xi(1-s)}{s}\mathcal{L}\left\{bN_h - \frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D}S_1 - \varepsilon\tau_1 S_1 - \rho S_1 - \alpha S_1\right\} \quad (4.3.1)$$

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$$\mathcal{L}\{S_2(t)\} = \frac{S_2(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\left\{\rho S_1 - \frac{\beta_2\left(\sigma_1 I_1 + \sigma_2 I_2\right)}{N_h - I_D} S_2 - \varepsilon\tau_2 S_2 - \alpha S_2\right\} \qquad (4.3.2)$$

$$\mathcal{L}\{V(t)\} = \frac{V(0)}{s} + \frac{s+\xi(1-s)}{s}\mathcal{L}\{\varepsilon\tau_1 S_1 + \varepsilon\tau_2 S_2 - \omega V - \alpha V\} \qquad (4.3.3)$$

$$\mathcal{L}\{E(t)\} = \frac{E(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\left\{\frac{\beta_1(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_1 + \frac{\beta_2(\sigma_1 I_1 + \sigma_2 I_2)}{N_h - I_D} S_2 - \sigma_1\delta E - \sigma_2(1-\delta)E - \alpha E\right\} \qquad (4.3.4)$$

$$\mathcal{L}\{I_1(t)\} = \frac{I_1(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\{\sigma_1\delta E - \gamma_1 I_1 - \theta_1 I_1 - \alpha I_1 - \eta I_1\} \qquad (4.3.5)$$

$$\mathcal{L}\{I_2(t)\} = \frac{I_2(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\{\sigma_2(1-\delta)E - \gamma_2 I_2 - \theta_2 I_2 - \alpha I_2 - \eta I_2\} \qquad (4.3.6)$$

$$\mathcal{L}\{I_D(t)\} = \frac{I_D(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\{\gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_D - \alpha R\} \qquad (4.3.7)$$

$$\mathcal{L}\{R(t)\} = \frac{R(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\{\theta_1 I_1 + \theta_2 I_2 - \gamma_3 I_D - \alpha I_D - \eta I_D\} \qquad (4.3.8)$$

According to the Adomian decomposition method, the solution will be in the following series type

$$S_1(t) = \sum_{n=0}^{\infty} S_{1n}(t),\ S_2(t) = \sum_{n=0}^{\infty} S_{2n}(t),\ V(t) = \sum_{n=0}^{\infty} V_n(t),\ E(t) = \sum_{n=0}^{\infty} E_n(t),\ I_1(t) = \sum_{n=0}^{\infty} I_{1n}(t),\ I_2(t) = \sum_{n=0}^{\infty} I_{2n}(t)\ I_D(t) = \sum_{n=0}^{\infty} I_{Dn}(t)(t),\quad \mathrm{R}(t) = \sum_{n=0}^{\infty} R_n(t) \qquad (4.4)$$

The non- linear term involved in the model are $I_1(t)S_1(t)$, $I_2(t)S_1(t)$, $I_1(t)S_2(t)$, $I_2(t)S_2(t)$. These are decomposed by Adomian Decomposition polynomial as

$$I_1(t)S_1(t) = \sum_{k=0}^{\infty} A_{1n},\ I_2(t)S_1(t) = \sum_{k=0}^{\infty} A_{2n},\ I_1(t)S_2(t) = \sum_{k=0}^{\infty} A_{3n},\ I_2(t)S_2(t) = \sum_{k=0}^{\infty} A_{4n} \qquad (4.5)$$

Where A_n is Adomian polynomial defined as:

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$$A_n = \frac{1}{\Gamma(k+1)}\frac{d^k}{dh^k}\left[\sum_{j=0}^{k} h^k S_j \sum_{j=0}^{k} h^k I_j\right]_{h=0} \qquad (4.6)$$

$$A_0 = S_0 I_0\,, \quad A_1 = S_1 I_0 + S_0 I_1, \quad A_2 = S_2 I_0 + S_1 I_1 + S_0 I_2, \quad A_3 = S_3 I_0 + S_2 I_1 + S_1 I_2 + S_0 I_3, \ . \quad . \quad . \qquad (4.7)$$

Applying equation (4.4)-(4.7) into the system (4.3.1)-(4.3.8), we have

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} S_{1n}(t)\right\} = \frac{S_1(0)}{s} + \frac{s+\xi(1-s)}{s}\mathcal{L}\left\{bN_h - -\frac{\beta_1\sigma_1 \sum_{n=0}^{\infty} A_{1n}}{N_h - \sum_{n=0}^{\infty} I_{Dn}} - \frac{\beta_1\sigma_2 \sum_{n=0}^{\infty} A_{2n}}{N_h - \sum_{n=0}^{\infty} I_{Dn}} - (\varepsilon\tau_1 + \rho + \alpha)\sum_{k=0}^{\infty} S_{1n}\right\} \qquad (4.8.1)$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} S_{2n}(t)\right\} = \frac{S_2(0)}{s} + \frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\rho\sum_{k=0}^{\infty} S_{1n} - \frac{\beta_2\sigma_1 \sum_{k=0}^{\infty} A_{3k}}{N_h - \sum_{k=0}^{\infty} I_{Dk}} - \frac{\beta_2\sigma_2 \sum_{k=0}^{\infty} A_{4k}}{N_h - \sum_{k=0}^{\infty} I_{Dk}} - (\varepsilon\tau_2 + \alpha)\sum_{k=0}^{\infty} S_{2k}\right\} \qquad (4.8.2)$$

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$$\mathcal{L}\left\{\sum_{n=0}^{\infty} V_n(t)\right\} = \frac{V(0)}{s} + \frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\varepsilon\tau_1\sum_{n=0}^{\infty} S_{1n} + \varepsilon\tau_2\sum_{n=0}^{\infty} S_{2n} - (\omega + \alpha)\sum_{k=0}^{\infty} V_n\right\} \quad (4.8.3)$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} E_n(t)\right\} = \frac{E(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\left\{\frac{\beta_1\sigma_1\sum_{n=0}^{\infty} A_{1n}}{N_h - \sum_{k=0}^{\infty} I_{Dk}} + \frac{\beta_1\sigma_2\sum_{n=0}^{\infty} A_{2n}}{N_h - \sum_{k=0}^{\infty} I_{Dk}} + \frac{\beta_2\sigma_1\sum_{k=0}^{\infty} A_{3k}}{N_h - \sum_{k=0}^{\infty} I_{Dk}} + \frac{\beta_2\sigma_2\sum_{k=0}^{\infty} A_{4k}}{N_h - \sum_{k=0}^{\infty} I_{Dk}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)\sum_{k=0}^{\infty} E_k\right\} \quad (4.8.4)$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} I_{1n}(t)\right\} = \frac{I_1(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\left\{\sigma_1\delta\sum_{k=0}^{\infty} E_k - \left(\gamma_1 + \theta_1 + \alpha + \eta\right)\sum_{k=0}^{\infty} I_{1k}\right\} \quad (4.8.5)$$

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$$\mathcal{L}\left\{\sum_{n=0}^{\infty} I_{2n}(t)\right\} = \frac{I_2(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\left\{\sigma_2(1-\delta)\sum_{n=0}^{\infty} E_n - (\gamma_2+\theta_2+\alpha+\eta)\sum_{n=0}^{\infty} I_{2n}\right\} \quad (4.8.6)$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} I_{Dn}(t)\right\} = \frac{I_D(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\left\{\theta_1\sum_{k=0}^{\infty} I_{1k} + \theta_2\sum_{k=0}^{\infty} I_{2k} - (\gamma_3+\alpha+\eta)\sum_{k=0}^{\infty} I_{Dk}\right\} \quad (4.8.7)$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} R_n(t)\right\} = \frac{R(0)}{s} + \frac{s+\xi(1-s)}{s}\ \mathcal{L}\left\{\gamma_1\sum_{n=0}^{\infty} I_{1n} + \gamma_2\sum_{n=0}^{\infty} I_{2n} + \gamma_3\sum_{n=0}^{\infty} I_{Dn} - \alpha\sum_{n=0}^{\infty} R_n\right\} \quad (4.8.8)$$

Using initial value condition, $I_D(0) = I_{D0}, R(0) = R_0, I_2(0) = I_{20} I_1(0) = I_{10}, E(0) = E_0, V(t) = V_0$, $S_2(0) = S_{20}, S_1(t) = S_{10}$, matching the items on both sides of (4.8.1) –(4.8.8) and applying $I_{1k}(t)S_{1k}(t) = A_{1k},\ I_{2k}(t)S_{1k}(t) = A_{2k},\ I_{1k}(t)S_{2k}(t) = A_{3k},\ I_{2k}(t)S_{2k}(t) = A_{4k}$ the general term of the model is given below

$$\sum_{n=0}^{\infty} S_{1(n+1)}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{bN_h - \frac{\beta_1\sigma_1 I_{1n}S_{1n}}{N_h - I_{Dn}} - \frac{\beta_1\sigma_2 I_{2n}S_{1n}}{N_h - I_{Dn}} - (\varepsilon\tau_1+\rho+\alpha)S_{1n}\right\}\right\} (4.9.1)$$

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$$\sum_{n=0}^{\infty} S_{2(n+1)}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\ \frac{1}{s^{\xi}}\mathcal{L}\left\{\rho S_{1n} - \frac{\beta_2\sigma_1 I_{1n}S_{2n}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{2n}S_{2n}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{2n}\right\}\right\} \quad (4.9.2)$$

$$\sum_{n=0}^{\infty} V_{n+1}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\varepsilon\tau_1 S_{1k} + \varepsilon\tau_2 S_{2n} - (\omega + \alpha)V_n\}\right\} \quad (4.9.3)$$

$$\sum_{n=0}^{\infty} E_{n+1}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\frac{\beta_1\sigma_1 I_{1n}S_{1n}}{N_h - I_{Dn}} + \frac{\beta_1\sigma_2 I_{2n}S_{1n}}{N_h - I_{Dn}} + \frac{\beta_2\sigma_1 I_{1n}S_{2n}}{N_h - I_{Dn}} + \frac{\beta_2\sigma_2 I_{2n}S_{2n}}{N_h - I_{Dn}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_n\right\}\right\} \quad (9.4)$$

$$\sum_{n=0}^{\infty} I_{1(n+1)}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_1\delta E_n - (\gamma_1 + \theta_1 + \alpha + \eta)I_{1n}\}\right\} \quad (4.9.5)$$

$$\sum_{n=0}^{\infty} I_{2(n+1)}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_2(1-\delta)E_n - (\gamma_2 + \theta_2 + \alpha + \eta)I_{2n}\}\right\} \quad (4.9.6)$$

$$\sum_{n=0}^{\infty} I_{D(n+1)}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\theta_1 I_{1n} + \theta_2 I_{2n} - (\gamma_3 + \alpha + \eta)I_{Dn}\}\right\} \quad (4.9.7)$$

$$\sum_{n=0}^{\infty} R_{n+1}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\gamma_1 I_{1n} + \gamma_2 I_{2n} + \gamma_3 I_{Dn} - \alpha R_n\}\right\} \quad (4.9.8)$$

Let $n = 0$

$$S_{11}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{bN_h - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1 + \rho + \alpha)S_{10}\right\}\right\} (4.10.1)$$

$$S_{21}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\ \frac{1}{s^{\xi}}\mathcal{L}\left\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\varepsilon\tau_2 + \alpha)S_{20}\right\}\right\} \quad (4.10.2)$$

$$V_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\varepsilon\tau_1 S_{10} + \varepsilon\tau_2 S_{20} - (\omega + \alpha)V_0\}\right\} \quad (4.10.3)$$

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$$E_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} + \frac{\beta_1\sigma_2 I_{2k}S_{1n}}{N_h - I_{D0}} + \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{D0}} + \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_n\right\}\right\} \quad (4.10.4)$$

$$I_{11}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\right\} \quad (4.10.5)$$

$$I_{21}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\}\right\} \quad (4.10.6)$$

$$I_{D1}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3 + \alpha + \eta)I_{D0}\}\right\} \quad (4.10.7)$$

$$R_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\gamma_1 I_{10} + \gamma_2 I_{20} + \gamma_3 I_{D0} - \alpha R_0\}\right\} \quad (4.10.8)$$

$$S_{11}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{bN_h - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1 + \rho + \alpha)S_{10}\right\}\right\} (4.10.1)$$

$$S_{21}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\frac{1}{s^{\xi}}\mathcal{L}\left\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{20}\right\}\right\} \quad (4.10.2)$$

$$V_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\varepsilon\tau_1 S_{10} + \varepsilon\tau_2 S_{20} - (\omega + \alpha)V_0\}\right\} \quad (4.10.3)$$

$$E_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} + \frac{\beta_1\sigma_2 I_{2k}S_{1n}}{N_h - I_{D0}} + \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{D0}} + \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_n\right\}\right\} \quad (4.10.4)$$

$$I_{11}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\right\} \quad (4.10.5)$$

$$I_{21}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\}\right\} \quad (4.10.6)$$

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$$I_{D1}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3 + \alpha + \eta)I_{D0}\}\right\} \quad (4.10.7)$$

$$R_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\gamma_1 I_{10} + \gamma_2 I_{20} + \gamma_3 I_{D0} - \alpha R_0\}\right\} \quad (4.10.8)$$

Applying the inverse Laplace transform, we obtain

$$S_{11}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{bN_h - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1 + \rho + \alpha)S_{10}\right\}\right\} (4.10.1)$$

$$S_{21}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\ \frac{1}{s^{\xi}}\mathcal{L}\left\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{20}\right\}\right\} \quad (4.10.2)$$

$$V_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\varepsilon\tau_1 S_{10} + \varepsilon\tau_2 S_{20} - (\omega + \alpha)V_0\}\right\} \quad (4.10.3)$$

$$E_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} + \frac{\beta_1\sigma_2 I_{2k}S_{1n}}{N_h - I_{D0}} + \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{D0}} + \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_n\right\}\right\} \quad (4.10.4)$$

$$I_{11}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\right\} \quad (4.10.5)$$

$$I_{21}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\ \mathcal{L}\{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\}\right\} \quad (4.10.6)$$

$$I_{D1}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3 + \alpha + \eta)I_{D0}\}\right\} \quad (4.10.7)$$

$$R_1(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\gamma_1 I_{10} + \gamma_2 I_{20} + \gamma_3 I_{D0} - \alpha R_0\}\right\} \quad (4.10.8)$$

$$S_{11}(t) = \left\{bN_h - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1 + \rho + \alpha)S_{10}\right\}\{1 + \xi(t-1)\} \quad (4.11.1)$$

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$$S_{21}(t) = \left\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{20}\right\}\{1 + \xi(t-1)\} \quad (4.11.2)$$

$$V_1(t) = \{\varepsilon\tau_1 S_{10} + \varepsilon\tau_2 S_{20} - (\omega + \alpha)V_0\}\{1 + \xi(t-1)\} \quad (4.11.3)$$

$$E_1(t) = \left\{\frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} + \frac{\beta_1\sigma_2 I_{2k}S_{1n}}{N_h - I_{D0}} + \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{D0}} + \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_0\right\}\{1 + \xi(t-1)\} \quad (4.11.4)$$

$$I_{11}(t) = \{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\{1 + \xi(t-1)\} \quad (4.11.5)$$

$$I_{21}(t) = \{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\}\{1 + \xi(t-1)\} \quad (4.11.6)$$

$$I_{D1}(t) = \{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3 + \alpha + \eta)I_{D0}\}\{1 + \xi(t-1)\} \quad (4.11.7)$$

$$R_1(t) = \{\gamma_1 I_{10} + \gamma_2 I_{20} + \gamma_3 I_{D0} - \alpha R_0\}\{1 + \xi(t-1)\} \quad (4.11.8)$$

Let $n = 1$

$$S_{12}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{bN_h - \frac{\beta_1\sigma_1 I_{11}S_{11}}{N_h - I_{D1}} - \frac{\beta_1\sigma_2 I_{21}S_{11}}{N_h - I_{D1}} - (\varepsilon\tau_1 + \rho + \alpha)S_{11}\right\}\right\} \quad (4.12.1)$$

$$S_{22}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\rho S_{11} - \frac{\beta_2\sigma_1 I_{1n}S_{21}}{N_h - I_{D1}} - \frac{\beta_2\sigma_2 I_{21}S_{21}}{N_h - I_{D1}} - (\varepsilon\tau_2 + \alpha)S_{21}\right\}\right\} \quad (4.12.2)$$

$$V_2(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\varepsilon\tau_1 S_{11} + \varepsilon\tau_2 S_{21} - (\omega + \alpha)V_1\}\right\} \quad (4.12.3)$$

$$E_2(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\left\{\frac{\beta_1\sigma_1 I_{1n}S_{1n}}{N_h - I_{Dn}} + \frac{\beta_1\sigma_2 I_{2n}S_{1n}}{N_h - I_{Dn}} + \frac{\beta_2\sigma_1 I_{1n}S_{2n}}{N_h - I_{Dn}} + \frac{\beta_2\sigma_2 I_{2n}S_{2n}}{N_h - I_{Dn}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_n\right\}\right\} \quad (4.12.4)$$

$$I_{12}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_1\delta E_1 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{11}\}\right\} \quad (4.12.5)$$

$$I_{22}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\sigma_2(1-\delta)E_1 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{21}\}\right\} \quad (4.12.6)$$

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$$I_{D2}(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\theta_1 I_{11} + \theta_2 I_{21} - (\gamma_3+\alpha+\eta)I_{D1}\}\right\} \qquad (4.12.7)$$

$$R_2(t) = \mathcal{L}^{-1}\left\{\frac{s+\xi(1-s)}{s}\mathcal{L}\{\gamma_1 I_{11} + \gamma_2 I_{21} + \gamma_3 I_{D1} - \alpha R_1\}\right\} \qquad (4.12.8)$$

$$\begin{aligned} S_{12}(t) = \Bigg\{ & bN_h \\ & - \left\{\frac{1}{N_h - [\{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3+\alpha+\eta)I_{D0}\}\{1+\xi(t-1)\}]}[\beta_1\sigma_1\{\{\sigma_1\delta E_0\right. \\ & - (\gamma_1+\theta_1+\alpha+\eta)I_{10}\}\{1+\xi(t-1)\}\} \\ & \left. - \beta_1\sigma_2\{\{\sigma_2(1-\delta)E_0 - (\gamma_2+\theta_2+\alpha+\eta)I_{20}\}\{1+\xi(t-1)\}\}] - (\varepsilon\tau_1+\rho+\alpha)\right\}\Bigg\{bN_h \\ & - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1+\rho+\alpha)S_{10}\Bigg\}\{1+\xi(t-1)\}\Bigg\}\{1+\xi(t-1)\} \quad 4.13.1 \end{aligned}$$

$$\begin{aligned} S_{22}(t) = = \Bigg\{ & \rho\left\{bN_h - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1+\rho+\alpha)S_{10}\right\} \\ & - \frac{1}{N_h - \{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3+\alpha+\eta)I_{D0}\}\{1+\xi(t-1)\}}[\beta_2\sigma_1\{\sigma_1\delta E_0 \\ & - (\gamma_1+\theta_1+\alpha+\eta)I_{10}\} - \beta_2\sigma_2\{\sigma_2(1-\delta)E_0 - (\gamma_2+\theta_2+\alpha+\eta)I_{20}\}] - (\varepsilon\tau_2+\alpha)\Bigg\}\Bigg\{\rho S_{10} \\ & - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2+\alpha)S_{20}\Bigg\}\{1+\xi(t-1)\}\{1+\xi(t-1)\} \qquad (4.13.2) \end{aligned}$$

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$$V_2(t) = \Big\{\varepsilon\tau_1\Big\{bN_h - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1 + \rho + \alpha)S_{10}\Big\} + \varepsilon\tau_2\Big\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{20}\Big\} - (\omega + \alpha)\{\varepsilon\tau_1 S_{10} + \varepsilon\tau_2 S_{20} - (\omega + \alpha)V_0\}\Big\}\{1 + \xi(t-1)\}\{1 + \xi(t-1)\} \quad (4.13.3)$$

$$E_2(t) = \Big\{\frac{1}{N_h - \{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3 + \alpha + \eta)I_{D0}\}\{1 + \xi(t-1)\}}[\beta_1\sigma_1\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\Big\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{20}\Big\} + \beta_1\sigma_2\{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\}\Big\{bN_h - \frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} - \frac{\beta_1\sigma_2 I_{20}S_{10}}{N_h - I_{D0}} - (\varepsilon\tau_1 + \rho + \alpha)S_{10}\Big\} + \beta_2\sigma_1\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\Big\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{20}\Big\} + \beta_2\sigma_2\{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\}\Big\{\rho S_{10} - \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{Dn}} - \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{Dn}} - (\varepsilon\tau_2 + \alpha)S_{20}\Big\}]\{1 + \xi(t-1)\} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)\Big\{\frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} + \frac{\beta_1\sigma_2 I_{2k}S_{1n}}{N_h - I_{D0}} + \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{D0}} + \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_n\Big\}\{1 + \xi(t-1)\}\Big\}\{1 + \xi(t-1)\} \quad (4.13.4)$$

$$I_{12}(t) = \Big\{\sigma_1\delta\Big\{\frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} + \frac{\beta_1\sigma_2 I_{2k}S_{1n}}{N_h - I_{D0}} + \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{D0}} + \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_0\Big\} - (\gamma_1 + \theta_1 + \alpha + \eta)\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\Big\}\{1 + \xi(t-1)\}\{1 + \xi(t-1)\} \quad (4.13.5)$$

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$$I_{22}(t) = \left\{\sigma_2(1-\delta)\left\{\frac{\beta_1\sigma_1 I_{10}S_{10}}{N_h - I_{D0}} + \frac{\beta_1\sigma_2 I_{2k}S_{1n}}{N_h - I_{D0}} + \frac{\beta_2\sigma_1 I_{10}S_{20}}{N_h - I_{D0}} + \frac{\beta_2\sigma_2 I_{20}S_{20}}{N_h - I_{D0}} - (\sigma_1\delta + \sigma_2(1-\delta) + \alpha)E_0\right\} - (\gamma_2 + \theta_2 + \alpha + \eta)\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\right\}\{1 + \xi(t-1)\}\{1 + \xi(t-1)\} \qquad (4.13.6)$$

$$I_{D2}(t) = \{\theta_1\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\}\{1 + \xi(t-1)\} + \theta_2\{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\}\{1 + \xi(t-1)\} - (\gamma_3 + \alpha + \eta)\{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3 + \alpha + \eta)I_{D0}\}\}$$

$$R_2(t) = \{\gamma_1\{\sigma_1\delta E_0 - (\gamma_1 + \theta_1 + \alpha + \eta)I_{10}\} + \gamma_2\{\sigma_2(1-\delta)E_0 - (\gamma_2 + \theta_2 + \alpha + \eta)I_{20}\} + \gamma_3\{\theta_1 I_{10} + \theta_2 I_{20} - (\gamma_3 + \alpha + \eta)I_{D0}\} - \alpha\{\gamma_1 I_{10} + \gamma_2 I_{20} + \gamma_3 I_{D0} - \alpha R_0\}\}\{1 + \xi(t-1)\}\{1 + \xi(t-1)\} \qquad (4.13.8)$$

Hence the required solution

$$S_1(t) = S_{10}(t) + S_{11}(t) + S_{12}(t) + \ldots$$

$$S_2(t) = S_{20}(t) + S_{21}(t) + S_{22}(t) + \ldots$$

$$V(t) = V_0(t) + V_1(t) + V_2(t) + \ldots$$

$$E(t) = E_0(t) + E_1(t) + E_2(t) + \ldots$$

$$I_1(t) = I_{10}(t) + I_{11}(t) + I_{12}(t) + \ldots$$

$$I_2(t) = I_{20}(t) + I_{21}(t) + I_{22}(t) + \ldots$$

$$I_D(t) = I_{D0}(t) + I_{D1}(t) + I_{D2}(t) + \ldots$$

$$R(t) = R_0(t) + R_1(t) + R_2(t) + \ldots$$

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Summary

Diphtheria remains a re-emerging public health concern despite vaccination programs, with children and adults exhibiting different levels of susceptibility and transmission potential. Classical integer-order models often fail to capture the **memory effects** inherent in disease transmission, such as waning immunity and delayed intervention impact, and few models incorporate **age structure**, which is crucial for diphtheria dynamics. Motivated by these limitations, this study develops a novel **age-structured fractional-order model** of diphtheria using the **Caputo–Fabrizio derivative**, which accounts for non-locality and memory effects in disease spread.

The model stratifies the population into susceptible children, susceptible adults, vaccinated, exposed, infectious children, infectious adults, isolated infectious, and recovered individuals. Rigorous analysis establishes the **existence and uniqueness** of solutions via the contraction mapping principle and proves **Ulam–Hyers stability**, confirming the robustness of the system under perturbations. To obtain approximate analytic–numerical solutions, the **Laplace Adomian Decomposition Method (LADM)** is applied, yielding a rapidly convergent series representation. Results show that the **fractional order significantly alters outbreak intensity and timing**, reflecting the role of memory in diphtheria persistence and control.

This work achieves three key outcomes: (i) the formulation of a new **fractional-order, age-structured diphtheria model**; (ii) provision of rigorous mathematical guarantees for solution behavior; and (iii) demonstration of efficient analytic–numerical solutions through LADM. The study contributes to knowledge by introducing a more realistic modeling framework that integrates age heterogeneity and memory effects, offering deeper epidemiological insights and practical guidance for sustaining vaccination and isolation strategies in diphtheria control.

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